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Ultralocal fields and their relevance for reparametrization-invariant quantum field theory

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Abstract

Reparametrization-invariant theories have a vanishing Hamiltonian and enforce their dynamics through a constraint. We specifically choose the Dirac procedure of quantization before the introduction of constraints. Consequently, for field theories, and prior to the introduction of any constraints, it is argued that the original field operator representation should be ultralocal in order to remain totally unbiased toward those field correlations that will be imposed by the constraints. It is shown that relativistic free and interacting theories can be completely recovered starting from ultralocal representations followed by a careful enforcement of the appropriate constraints. In so doing all unnecessary features of the original ultralocal representation disappear.

The present discussion is germane to a recent theory of affine quantum gravity in which ultralocal field representations have been invoked before the imposition of constraints.

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1. Introduction

Basic classical model. A variety of theories enjoy invariance under reparametrization of the time variable, and this feature represents one aspect of a favoured goal of achieving invariance under arbitrary coordinate transformations. We have in mind theories as complex as general relativity or as simple as a single-degree-of-freedom dynamical system expressed in a reparametrization-invariant formulation. For a single-degree-of-freedom system, let the initial variables of the canonical formulation be called p and q . We note that the physics of the situation typically dictates the dimensions of the variables p and q , and occasionally the dimensions of the momentum and coordinate variables will be of interest. We generally describe a simple dynamical system by the action functional

$$I = \int [p \dot{q} - H(p, q)] dt$$

where $\dot{q} = dq/dt$, and the equations of motion it gives rise to, namely

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

subject to suitable boundary data.

By a reparametrization-invariant formulation we have in mind the alternative action functional

$$I' = \int \{p (dq/d\tau) + s (dt/d\tau) - \lambda(\tau)[s + H(p, q)]\} d\tau.$$

In going from I to I' we have introduced a new independent variable τ , elevated the former independent variable t to a dependent (dynamical) variable and introduced its conjugate variable s . Additionally, we note that in the new form the Hamiltonian vanishes, while in its place we have a Lagrange multiplier variable $\lambda(\tau)$ which enforces the (first-class) constraint $s + H(p, q) = 0$. Based on these facts, the equations of motion that arise from I' read as

$$\frac{dq}{d\tau} = \lambda \frac{\partial H}{\partial p} \quad \frac{dp}{d\tau} = -\lambda \frac{\partial H}{\partial q} \quad \frac{dt}{d\tau} = \lambda \quad \frac{ds}{d\tau} = 0$$

along with the constraint $s + H(p, q) = 0$. From these equations we deduce that $dq/dt = (dt/d\tau)(\partial H/\partial p)$, etc; that is,

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

exactly as before.

Observe, in the former case, that the dynamics arose from the *Hamiltonian*; in the latter case, the dynamics arose from enforcing a constraint, that is, it arose from the *kinematics*.

Basic quantum model. A quantum analysis of reparametrization invariance follows the spirit of the preceding discussion. The initial quantum theory assumes canonical, self-adjoint operators P and Q , with $[Q, P] = i$ (in units where $\hbar = 1$), and a Hamiltonian operator $\mathcal{H} = \mathcal{H}(P, Q)$ all acting in a Hilbert space \mathfrak{H} . The operator equations of motion are given by

$$\frac{dQ(t)}{dt} = i[\mathcal{H}, Q(t)] \quad \frac{dP(t)}{dt} = i[\mathcal{H}, P(t)]$$

and expectations of interest are given, say, by

$$\langle \psi | Q(t_1) Q(t_2) \cdots Q(t_n) | \psi \rangle$$

etc, for a general $|\psi\rangle \in \mathfrak{H}$.

As another characterization of the theory, we may also employ a coherent state representation [1]. To that end let us introduce

$$|p, q\rangle \equiv e^{-iqP} e^{ipQ} |\eta\rangle$$

where $|\eta\rangle$ denotes a suitable normalized fiducial vector which we generally choose as an oscillator ground state defined by $(\Omega Q + iP)|\eta\rangle = 0$. In that case,

$$\langle p'', q' | p', q' \rangle = \exp \left\{ \frac{1}{2} i(p'' + p')(q'' - q') - \frac{1}{4} [\Omega^{-1}(p'' - p')^2 + \Omega(q'' - q')^2] \right\}.$$

Here, $\Omega > 0$ is a free parameter of the coherent state representation and may be chosen arbitrarily. Under normal circumstances, a parameter such as Ω is necessary on dimensional grounds, but even if it is dimensionless it is still a freedom in the coherent state representation. For simplicity, it is often convenient to set $\Omega = 1$; however, we shall not do so. It is important to observe that the coherent state overlap $\langle p'', q'' | p', q' \rangle$ defines a positive-definite function which we may choose as a *reproducing kernel* and use it to define the associated *reproducing*

kernel Hilbert space [2]. With regard to dynamics for this system, we may study the propagator implicitly given by

$$\langle p'', q'' | e^{-i(t''-t')\mathcal{H}} | p', q' \rangle.$$

Hereafter, we shall principally focus on coherent state formulations.

For the reparametrization-invariant formulation, on the other hand, the quantization procedure is somewhat different. Adopting Dirac's method [3], we proceed as follows. Let P, Q and S, T denote two independent canonical pairs with $[S, T] = i$ as the only additional nonvanishing commutator. Thus the Hilbert space \mathfrak{H}' is now that for a two-degree-of-freedom problem. To reduce that space to one appropriate to a single degree of freedom, we (ideally) select out a subspace $\mathfrak{H}_{\text{phys}} \subset \mathfrak{H}'$ by the constraint condition

$$[S + \mathcal{H}(P, Q)] |\psi\rangle_{\text{phys}} = 0.$$

It is especially convenient to choose a coherent state approach in which to enforce this kind of constraint [4]. To that end we introduce the coherent state

$$|p, q, s, t\rangle = e^{-iqP} e^{ipQ} e^{-irS} e^{isT} |\eta'\rangle = |p, q\rangle \otimes |s, t\rangle$$

in the Hilbert space \mathfrak{H}' . The overlap of two such coherent states is given (with F an evident normalization factor) by

$$\begin{aligned} \langle p'', q'', s'', t'' | p', q', s', t' \rangle &= \langle p'', q'' | p', q' \rangle \langle s'', t'' | s', t' \rangle \\ &= F \iint e^{-(k-p'')^2/2\Omega - (u-s'')^2/2\Lambda} e^{i(q''-q')k + i(t''-t')u} e^{-(k-p')^2/2\Omega - (u-s')^2/2\Lambda} dk du. \end{aligned}$$

We impose the constraint in this expression in the form

$$\begin{aligned} \langle p'', q'', s'', t'' | \delta(S + \mathcal{H}(P, Q)) | p', q', s', t' \rangle \\ \equiv \lim_{\delta \rightarrow 0} (2\delta)^{-1} \langle p'', q'', s'', t'' | \mathbb{E}([S + \mathcal{H}(P, Q)]^2 \leq \delta^2) | p', q', s', t' \rangle \end{aligned}$$

where \mathbb{E} is a projection operator onto the spectral interval $-\delta \leq [S + \mathcal{H}(P, Q)] \leq \delta$, which, up to a multiplicative factor, leads to

$$\begin{aligned} \langle\langle p'', q'', s'', t'' | p', q', s', t' \rangle\rangle \\ \equiv \langle p', q', 0, 0 | e^{-(s''+\mathcal{H}(P, Q))^2/2\Lambda} e^{-i(t''-t')\mathcal{H}} e^{-(s'+\mathcal{H}(P, Q))^2/2\Lambda} | p', q', 0, 0 \rangle. \end{aligned}$$

At this point the variables s and t do not add to the span of the reduced Hilbert space. Or, to put it another way, the sets of states $\{|p, q, s, t\rangle\}$ and $\{|p, q, 0, 0\rangle\}$ span the same Hilbert space. Therefore, we can for example integrate out the variables s'' and s' without changing matters, which, with a suitable rescaling, then leads to the positive-definite function

$$\langle\langle p'', q'', t'' | p', q', t' \rangle\rangle \equiv \langle p'', q'', 0, 0 | e^{-i(t''-t')\mathcal{H}} | p', q', 0, 0 \rangle.$$

Finally, we may simply identify the last result with the same expression as obtained previously, that is,

$$\langle\langle p'', q'', t'' | p', q', t' \rangle\rangle = \langle p'', q'' | e^{-i(t''-t')\mathcal{H}} | p', q' \rangle$$

completing the argument that the reparametrization formulation leads to the same result as the original formulation. This example also illustrates how 'time' may emerge within a reparametrization-invariant formulation.

Discussion. The foregoing examples have been limited to a single degree of freedom. It is clear, however, that an analogous discussion holds in the case of finitely many degrees of freedom. While the parameter Λ has disappeared from the final fully constrained theory, the parameter Ω of the coherent state representation remains. However, it merely corresponds to an inherent freedom in the representation that has no physical consequence whatsoever.

The situation changes, however, when an infinite number of degrees of freedom are present. As is well known, and unlike the situation for finitely many degrees of freedom, fields that satisfy canonical commutation relations possess uncountably many inequivalent irreducible representations, and it is imperative that one choose the correct one in order for a field theory to make sense [5]. Prior to the introduction of the constraint(s), there is in principle no information from which to make a proper choice of field operator representation. In fact, before any constraints have been introduced, one must essentially turn a blind eye toward what constraints are to come and initially use a representation that is as neutral toward the situation as is possible. With that thought in mind, we are led to propose that the only neutral field operator representations that should be considered are *ultralocal representations*. Ultralocal representations are defined as those that have *no* correlations between fields at distinct spatial points, i.e. fields whose values at distinct spatial points are statistically independent. Any correlations between fields at distinct spatial points that are required by the physics will be introduced by the constraints that are imposed later. (Alternative quantization procedures which reduce first and quantize second can lead to erroneous results [6].)

Having argued that, before the introduction of constraints, the initial field operator representation should be ultralocal, leads to a possible dilemma since the ultralocal field operator representations are surely inequivalent to the desired field operator representations that are expected once the constraints are fully enforced. The question arises as to whether this inequivalence, or more descriptively, this huge distance between representations can be bridged, and, even if it can, will there be any unwelcome residue remaining of the original and unphysical ultralocal representation. Happily, as we shall learn, it is entirely possible to start with an ultralocal representation, as an unbiased starting point requires, and, nevertheless, emerge with the proper representation for the problem at hand—and, moreover, no unwelcome residue of the original representation remains. In the sense described, ultralocal field theories appear to have found genuine physical applications in the quantization of reparametrization-invariant field theories.

Indeed, elsewhere [7], we have already discussed the important role ultralocal fields play in the quantization of the gravitational field by affine field variables before any constraints are introduced; this paper may be considered as an adjunct to this same programme for quantum gravity.

In principle, there is one marked advantage in starting with a reparametrization-invariant formulation and the associated ultralocal representations of the fields that are involved. The advantage of doing so arises from the fact that *all ultralocal representations can be classified, and they already have been rather thoroughly investigated* [8] (see section 3 below).

Remaining sections. In the following section, section 2, we shall extend the analysis of the present section to several different field theories: relativistic free fields, and relativistic ϕ_2^4 and ϕ_3^4 models. In section 3, we shall review the most general ultralocal field operator representations, briefly revisit our discussion in section 2 and just mention the case of ultralocal model fields themselves and how they fit into the general scheme.

Speaking more generally, we will argue in the following sections that even though ultralocal fields are initially used, the usual (and generally nonultralocal) results still emerge from the reparametrization-invariant formulation after the constraints have been fully enforced.

2. Relativistic fields

2.1. Conventional free fields

Classical theory. A classical relativistic free field $\phi = \phi(x, t)$ (and momentum $\pi = \pi(x, t)$) of mass m in (say) four spacetime dimensions may be described by the action functional

$$I = \int dt \int d^3x \left\{ \pi \dot{\phi} - \frac{1}{2} [\pi^2 + (\nabla\phi)^2 + m^2\phi^2] \right\}$$

which leads to the equations of motion $\partial\phi/\partial t \equiv \dot{\phi} = \pi$ and

$$\dot{\pi} = \ddot{\phi} = \nabla^2\phi - m^2\phi.$$

Given suitable initial conditions, solution of these equations of motion follows conventional lines.

Quantum theory. Canonical quantization proceeds by introducing irreducible, locally self-adjoint field $\hat{\phi}(x)$ and momentum $\hat{\pi}(x)$ operators that satisfy the canonical commutation relations

$$[\hat{\phi}(x), \hat{\pi}(y)] = i\delta(x - y)$$

with all other commutators vanishing. Of all the (inequivalent) operator representations that fulfill the commutation relations, attention is focused on that one which admits

$$\mathcal{H} \equiv \frac{1}{2} \int : \{ \hat{\pi}(x)^2 + [\nabla\hat{\phi}(x)]^2 + m^2\hat{\phi}(x)^2 \} : d^3x$$

as a non-negative, self-adjoint operator, and for which $|0\rangle$ denotes a nondegenerate, normalized ground state with $\mathcal{H}|0\rangle = 0$. Here the colons $:$ signify normal ordering with respect to $|0\rangle$. The chosen representation of $\hat{\phi}(x)$ and $\hat{\pi}(x)$ depends on m and is unitarily inequivalent to any other representation defined for a value $m' \neq m$.

We adopt a coherent state representation for this example (see, e.g., [9]), and we therefore introduce

$$|\pi, \phi\rangle \equiv e^{i[\hat{\phi}(\pi) - \hat{\pi}(\phi)]} |0\rangle$$

where $\hat{\phi}(\pi) = \int \hat{\phi}(x)\pi(x) d^3x$ and $\hat{\pi}(\phi) = \int \hat{\pi}(x)\phi(x) d^3x$ are defined for smooth (test) functions π and ϕ . (We have adopted a different phase convention for the coherent states as compared to that in section 1, but this is of no significance.) In turn, the propagator in this representation is expressed by

$$\langle \pi'', \phi'' | e^{-i(t''-t')\mathcal{H}} | \pi', \phi' \rangle$$

and it has the explicit expression given by

$$\langle \pi'', \phi'' | e^{-i(t''-t')\mathcal{H}} | \pi', \phi' \rangle = N'' N' \exp \left[\int z''^*(k) e^{-i(t''-t')\omega(k)} z'(k) d^3k \right]$$

where in each case $z(k) \equiv (1/\sqrt{2})[\omega(k)^{1/2}\tilde{\phi}(k) + i\omega(k)^{-1/2}\tilde{\pi}(k)]$, and $\tilde{\pi}(k)$ and $\tilde{\phi}(k)$ are the Fourier transform of $\pi(x)$ and $\phi(x)$, respectively. The factors N'' and N' ensure normalization and are given, in each case, by

$$N = \exp \left[-\frac{1}{2} \int |z(k)|^2 d^3k \right]$$

and finally $\omega(k) \equiv (k^2 + m^2)^{1/2}$.

It is noteworthy that the mass parameter m labels *mutually distinct* Hilbert spaces with the property that any vector $|\psi; m\rangle \in \mathfrak{H}_m$ and any vector $|\chi; m'\rangle \in \mathfrak{H}_{m'}$, $m \neq m'$, are *orthogonal*.

Another way of saying this is that the field operator representations of $\hat{\phi}$ and $\hat{\pi}$ are unitarily inequivalent for any two mass values m and m' whenever $m \neq m'$. We stress these facts to emphasize that quantum theories of relativistic free fields are, for different mass values, fundamentally distinct from one another; if you have one of them, you definitely do *not* have any of the others!

2.2. Reparametrization-invariant free fields

Classical theory. In analogy with the simple example of section 1, we introduce the reparametrization-invariant action for the relativistic free field of mass m given by

$$I' = \int d\tau \left(s t^* + \int d^3x [\pi \phi^*] - \lambda \left\{ s + \frac{1}{2} \int d^3x [\pi^2 + (\nabla\phi)^2 + m^2\phi^2] \right\} \right)$$

where $t^* \equiv dt/d\tau$, $\phi^* \equiv \partial\phi/\partial\tau$ and $\lambda = \lambda(\tau)$ denotes a Lagrange multiplier function. It is clear that the equations of motion that follow from I' lead to the same equations of motion as follow from I , just as was the case for the simple example in section 1.

Quantum theory. Once again we introduce coherent states for the combined system given by

$$|\pi, \phi, s, t\rangle \equiv |\pi, \phi\rangle \otimes |s, t\rangle$$

where in the present case we adopt an ultralocal representation for the field operators since we assume prior ignorance of the constraints that are yet to be introduced. As a consequence, the overlap of two such coherent states is given by

$$\begin{aligned} \langle \pi'', \phi'', s'', t'' | \pi', \phi', s', t' \rangle &= L'' L' \exp \left[\int u''^*(x) u'(x) d^3x \right] \\ &\times \exp \left\{ -\frac{1}{4} [\Lambda^{-1}(s'' - s')^2 - 2i(s'' + s')(t'' - t') + \Lambda(t'' - t')^2] \right\} \end{aligned}$$

where in the present case

$$\begin{aligned} u(x) &\equiv (1/\sqrt{2})[M^{1/2}\phi(x) + iM^{-1/2}\pi(x)] \\ L &\equiv \exp \left[-\frac{1}{2} \int |u(x)|^2 d^3x \right] \end{aligned}$$

in both of the cases $u = u''$ and $u = u'$. Here M , $0 < M < \infty$, is a new parameter in the ultralocal representation which labels *inequivalent* field operator representations for $\hat{\pi}$ and $\hat{\phi}$, representations which in addition are inequivalent to all relativistic free field representations for any mass m . For the usual free fields that we are considering in this section, M must have the dimensions of 'mass', and thus M is necessary simply on dimensional grounds if for no other reason. Like the parameter Ω in the simple example of section 1, we are free to choose it; but in the present case, since it labels inequivalent field operator representations, it would appear that the choice of M is by no means benign.

In contrast to such appearances, we shall now show that one may start with *any* choice of $M > 0$ and emerge with the usual quantization of the relativistic free field for *any* (independent) choice of $m \geq 0$ (with, as usual, extra care needed when $m = 0$ if the space is one dimensional instead of three). In fact, when the constraint is fully enforced, we will find that *all traces of the arbitrary parameter M will disappear from the final result.*

As in the simple example of section 1, we now proceed to introduce the constraint. However, the operator form of the quantum constraint strongly depends on having the 'right' field operator representation. Stated otherwise, if we build

$$\mathcal{H} = \frac{1}{2} \int : \{ \hat{\pi}(x)^2 + [\nabla\hat{\phi}(x)]^2 + m^2\hat{\phi}(x)^2 \} : d^3x$$

out of the present field operator representation plus normal ordering with respect to the fiducial vector (or any vector for that matter), we will find that it is *not an operator*, or more precisely that it has only the zero vector in its domain. To proceed, let us first introduce a regularized form of the operators appearing in the constraint. To that end, let $\{h_n(x)\}_{n=1}^\infty$ denote a complete orthonormal set of real functions over the configuration space. For each $N \in \{1, 2, 3, \dots\}$, define the sequence of kernels

$$K_N(x, y) \equiv \sum_{n=1}^N h_n(x) h_n(y)$$

which have the property that, as $N \rightarrow \infty$, $K_N(x, y) \rightarrow \delta(x - y)$, in the sense of distributions. Next consider the regularized fields

$$\begin{aligned} \hat{\pi}_N(x) &\equiv \int K_N(x, y) \hat{\pi}(y) d^3y \\ \hat{\phi}_N(x) &\equiv \int K_N(x, y) \hat{\phi}(y) d^3y. \end{aligned}$$

Now build the regularized Hamiltonian operator

$$\mathcal{H}_N \equiv \frac{1}{2} \int : \{ \hat{\pi}_N(x)^2 + [\nabla \hat{\phi}_N(x)]^2 + m^2 \hat{\phi}_N(x)^2 \} : d^3x$$

which also involves normal ordering with respect to the fiducial vector. Observe, for $N < \infty$, that the operator \mathcal{H}_N involves only a finite number of degrees of freedom.

We are now in a position to enforce the regularized constraint by considering

$$\langle \pi'', \phi'', s'', t'' | \delta(S + \mathcal{H}_N) | \pi', \phi', s', t' \rangle$$

which, as in section 1, up to a finite coefficient, assumes the form

$$\langle \pi'', \phi'', 0, 0 | e^{-(s'' + \mathcal{H}_N)^2/2\Lambda} e^{-i(t'' - t')\mathcal{H}_N} e^{-(s' + \mathcal{H}_N)^2/2\Lambda} | \pi', \phi', 0, 0 \rangle.$$

In this form we again observe that the parameters s and t do not lead to any new vectors in the reduced Hilbert space. In other words, s and t no longer play a role in spanning the space of vectors. While this expression is well defined for all $N < \infty$, it will vanish identically in the limit $N \rightarrow \infty$ for any choice of m (because, in effect, $\mathcal{H}_N \rightarrow \infty$). That is one signal that the present (ultralocal) representation is *not* well suited to the relativistic free field. This fact is of course not surprising, but it is comforting to have an unambiguous analytic signal of this incompatibility.

It does not help to integrate out the variables s'' and s' . The result of that integration, apart from finite factors, is given by

$$\langle \pi'', \phi'', 0, 0 | e^{-i(t'' - t')\mathcal{H}_N} | \pi', \phi', 0, 0 \rangle$$

which is well defined and continuous in its arguments when $N < \infty$, but is *no longer* continuous in the time variable after taking the limit $N \rightarrow \infty$. Specifically, the limit is well defined when $t'' = t'$, but it is undefined when $t'' \neq t'$. This is yet another concrete signal that the ultralocal representation is not well suited to the relativistic free field.

As already stated, these unsuitable limits have emerged because we do not have the right field operator representation. However, we can fix that. Let us first make a selection of which relativistic free field we are interested in; i.e., let us choose a mass m . Then, by taking suitable linear combinations of the previous expression we can build a new expression given by

$$\langle \pi'', \phi''; N, m | e^{-i(t'' - t')\mathcal{H}_N} | \pi', \phi'; N, m \rangle$$

which involves coherent states that are *not* based on the original fiducial vector $|\eta\rangle$, but, instead, are based on the fiducial vector $|N, m\rangle$ which for the first N degrees of freedom is defined to be

the ground state of the operator \mathcal{H}_N , and is unchanged for the remaining degrees of freedom. We suppose further that \mathcal{H}_N is also normal ordered with respect to the vector $|N, m\rangle$. It is now clear that we can indeed take the limit $N \rightarrow \infty$ and obtain an expression which is continuous in the time variable in the limit. In this manner we have effectively passed from the wrong (ultralocal) representation of the field operators to the right (relativistic) representation of the field operators. After $N \rightarrow \infty$, the result becomes

$$\langle \pi'', \phi'' | e^{-i(t''-t')\mathcal{H}} | \pi', \phi' \rangle$$

where

$$\mathcal{H} = \frac{1}{2} \int : \{ \hat{\pi}(x)^2 + [\nabla \hat{\phi}(x)]^2 + m^2 \hat{\phi}(x)^2 \} : d^3x$$

is now well defined and coincides with the usual free field Hamiltonian. Therefore, the final propagator *completely coincides* with the expression we previously found with the same name. This argument demonstrates that the reparametrization-invariant formulation leads to the same result as the usual theory even though we started from an ultralocal formulation of the unconstrained theory. Observe, in addition, that all traces of the initial parameter M have disappeared in the final expression. In short, whatever M value had been originally chosen, it served only as a ‘place holder’ for the proper expression (effectively, $\omega(k)$, which it should be noted has the same dimensions as M) that emerged when the constraint was fully enforced.

There may well be some questions remaining about how one may attain the ‘suitable linear combinations’ to secure our desired result. To that end we offer the following rather picturesque description of how that may be accomplished. Consider the set of inner product quotients of a dense set of vector norms given by

$$S \equiv \left\{ \frac{\sum_{j,k=1}^J a_j^* a_k \langle \pi_j, \phi_j, 0, 0 | e^{-\mathcal{H}_N/\Lambda} | \pi_k, \phi_k, 0, 0 \rangle}{\sum_{j,k=1}^J a_j^* a_k \langle \pi_j, \phi_j, 0, 0 | \pi_k, \phi_k, 0, 0 \rangle} : J < \infty \right\}$$

where $\{a_j\}$ denote complex coefficients (not all zero), while $\{\pi_j\}$ and $\{\phi_j\}$ denote suitable fields. (The numerator of this quotient arises from the original expression when $s'' = s' = 0$ and $t'' = t'$; the denominator of this quotient arises after integrating out the variables s'' and s' when $t'' = t'$.) Now, as N becomes large, it will happen that most of the inner product quotients in the set S will become exponentially small, while some will not. The vectors whose quotients do not become small lie in the general direction in the pre-Hilbert space where the proper ground state has the largest overlap with the present set of vectors. One then takes the necessary linear combination of vectors, and any necessary limits, to change the fiducial vector to the unit vector that maximizes this overlap. Introducing a new set of coherent states that has the maximizing vector as its fiducial vector is a process that may be called *recentering the coherent states* or equivalently *recentering the reproducing kernel*. It is this procedure that ‘homes in’ on the right representation of the field operators for the relativistic free field, and ultimately leads to a suitable and continuous expression for the reproducing kernel for the relativistic free field. It is clear in this construction that in the limit all traces of the original ultralocal representation disappear and one is left only with the proper relativistic expression.

In summary, we note that regularization of the constraint and careful recentering of the coherent states as the regularization is removed are the key procedures in passing from the form of the expression before the constraint has been introduced to the form in which the constraint has been fully enforced. Equivalence between the usual and reparametrization-invariant formulations has thus been established, which was our goal.

2.3. Other relativistic fields

The preceding discussion was confined to relativistic free fields, but we may also consider interacting fields as well. In particular, the so-called ϕ_2^4 and ϕ_3^4 models have been rigorously constructed [10]. Both of these models admit canonical field and momentum operators that obey canonical commutation relations. In addition, both models admit unique ground states to their respective Hamiltonian operators, but in the present case the ground states are not Gaussian in a field diagonal representation as befit truly interacting theories. Thus the associated coherent state overlap function of interest in the present case is *not* quasi-free. Nevertheless, we can proceed exactly as before, namely, introduce cutoff fields and thereby a cutoff Hamiltonian (with a cutoff-dependent bare mass for ϕ_3^4), use that cutoff Hamiltonian to identify a vector approximating the ground state of the cutoff Hamiltonian and then recenter the coherent states about a new vector which is based on the ground state of the regularized Hamiltonian. Seeking the vector that maximizes the elements of the set S will again work in the present case as a means to identify the approximate ground state. Taking the appropriate limit as the regularization is removed will, just as in the free case, lead to a new reproducing kernel and thereby an associated reproducing kernel Hilbert space appropriate to the case at hand. Once again all traces of the ultralocal representation will disappear in the limit that the regularization is removed. Hence, we see that the procedures discussed above for the free field case will extend to handle those irreducible interacting fields that still satisfy a set of canonical commutation relations.

3. Most general ultralocal fields

In this section we want to discuss more general classes of ultralocal field operator representations than those already dealt with in the previous section. We also want to see how these more general representations fit into the kind of analysis that was carried out for the relativistic fields in section 1.

As a first example, we cite the expression for a single field, such as the field operator, independent of whether or not it satisfies the canonical commutation relations (in the usual sense) with another field. In particular, we note that the most general expression that respects ultralocality is given by

$$\langle \eta | e^{i\hat{\phi}(\pi)} | \eta \rangle = \exp \left(\int d^3x \left\{ ia(x)\pi(x) - \frac{1}{4}c(x)\pi(x)^2 + \int [e^{i\lambda\pi(x)} - 1 - i\lambda\pi(x)/(1 + \lambda^2)] d\sigma(\lambda; x) \right\} \right)$$

where $a(x)$ is an arbitrary real function, $c(x)$ is an arbitrary non-negative function and σ is a non-negative measure with the property that

$$\int [\lambda^2/(1 + \lambda^2)] d\sigma(\lambda; x) < \infty$$

for almost all x . We do not prove this expression here but rather refer the reader to [8], where a homogeneous form of this result is proved. It is straightforward to amend that argument to cover the inhomogeneous case.

In case $c(x) = 0$ for a set of nonzero measure, then it is generally necessary that $\int d\sigma(\lambda; x) = \infty$ in that spatial region in order that the smeared field $\hat{\phi}(\pi)$ has a purely continuous spectrum; however, we will not pursue technical issues of this sort here.

Canonical fields. As another class of examples, suppose that the field and momentum operators are both involved and that they satisfy conventional canonical commutation relations. Then the most general ultralocal representation is determined by

$$\langle \eta | e^{i[\hat{\phi}(\pi) - \hat{\pi}(\phi)]} | \eta \rangle = \exp \left(\int d^3x \left\{ ia(x)\pi(x) - ib(x)\phi(x) - \frac{1}{4}[c(x)\pi(x)^2 + d(x)\phi(x)^2] \right. \right. \\ \left. \left. + \int [e^{i\lambda\pi(x)} - 1 - i\lambda\pi(x)/(1 + \lambda^2)] d\sigma(\lambda; x) \right. \right. \\ \left. \left. + \int [e^{-i\gamma\phi(x)} - 1 + i\gamma\phi(x)/(1 + \gamma^2)] d\rho(\gamma; x) \right\} \right)$$

subject to the conditions that $a(x)$ and $b(x)$ are arbitrary real functions, $c(x)$ and $d(x)$ are positive definite and moreover $c(x)d(x) \geq 1$ holds for almost all x , and, finally, ρ is a non-negative measure that fulfills a similar condition as σ , namely

$$\int [\gamma^2/(1 + \gamma^2)] d\rho(\gamma; x) < \infty$$

for almost all x .

We remark that if either σ or ρ is nonzero, or if $c(x)d(x) > 1$ for a nonzero spatial measure, then the field and momentum operator representation is reducible rather than irreducible. If the expressions a , b , c , d , σ and ρ are independent of x , then the representations are homogeneous. Homogenous expressions generally suffice to deal with translation invariant constraints, such as the examples dealt with in section 2.

3.1. Relativistic free field, revisited

Let us consider the free field once again, and suppose we had chosen a different ultralocal representation to start with. In particular, suppose we chose

$$\langle \pi', \phi' | \pi', \phi' \rangle_a = \exp \left(\int d^3x \left(\frac{1}{2}i[\phi''(x)\pi'(x) - \pi''(x)\phi'(x)] \right. \right. \\ \left. \left. - \frac{1}{4}\{M^{-1}[\pi''(x) - \pi'(x)]^2 + M[\phi''(x) - \phi'(x)]^2\} + ia(x)[\phi''(x) - \phi'(x)] \right) \right)$$

which corresponds to an irreducible representation whatever generalized function is chosen for a . Recentring the coherent states proceeds as in section 2, for any a , and results in the proper relativistic coherent state overlap for any pre-given mass m . Summarizing, we observe that recentring the reproducing kernel brings us to the relativistic free field form for any choice of $a(x)$. Since that is the case, it follows that any suitable superposition of similar expressions over the generalized function a will result in the proper free field case. In particular, a local, normalized Gaussian superposition over $a(x)$, such as

$$\int \exp \left(i \int a(x)[\phi''(x) - \phi'(x)] d^3x \right) d\mu(a) = \exp \left(-\frac{1}{4}\tilde{M} \int [\phi''(x) - \phi'(x)]^2 d^3x \right)$$

leads to

$$\langle \pi'', \phi'' | \pi', \phi' \rangle \equiv \int \langle \pi'', \phi'' | \pi', \phi' \rangle_a d\mu(a) = \exp \left(\int d^3x \left(\frac{1}{2}i[\phi''(x)\pi''(x) - \pi''(x)\phi'(x)] \right. \right. \\ \left. \left. - \frac{1}{4}\{M^{-1}[\pi''(x) - \pi'(x)]^2 + M'[\phi''(x) - \phi'(x)]^2\} \right) \right).$$

In this expression, $M^{-1}M' = M^{-1}(M + \tilde{M}) > 1$, implying that the indicated ultralocal coherent state overlap corresponds to a reducible representation of the canonical commutation relations.

The result of this exercise shows that since the recentred reproducing kernel (after enforcing the constraint) is given, for any original $a(x)$, by the unique form for the relativistic free field, it follows that the superposition of such expressions also leads to the relativistic free field form. Stated otherwise, we observe that it is even possible to choose an initial ultralocal field operator representation that is *reducible*, and, nevertheless, after regularization and recentring of the coherent states, one emerges with the proper *irreducible* representation for the relativistic free field of mass m in which all traces of the original parameters M and M' ($> M$) disappear.

3.2. Ultralocal model fields

The last example we consider refers to the so-called (even) ultralocal field theories themselves [8]. These models only possess sharp time local field operators (and not sharp time local momentum operators), and thus we recall that

$$\langle \pi'' | \pi' \rangle = \langle \eta | e^{-i[\hat{\phi}(\pi'') - \hat{\phi}(\pi')]} | \eta \rangle = \exp \left(-b \int d^3x \int [1 - \cos\{\lambda[\pi''(x) - \pi'(x)]] c(\lambda)^2 d\lambda \right)$$

$c(-\lambda)^2 = c(\lambda)^2$, which is clearly of the general form indicated earlier in this section. We focus on the dimensional parameter b which appears here as a coefficient. It is noteworthy, for these models, that the precise value of b is *not* determined by the form of the Hamiltonian. In the spirit of this paper, the parameter b is a feature of the original ultralocal representation that does *not* disappear after the constraint regarding the dynamics is fully enforced. We draw from this example the conclusion that while it is possible that all traces of the original ultralocal representation may disappear once the constraints have been fully enforced, it is not always required to be the case. This precaution may be useful to keep in mind in studying further examples of reparametrization-invariant field theories and their quantization using ultralocal representations.

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